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SERIES SOLUTION OF UNSTEADY HEAT OR MASS TRANSFER TO A TRANSLATING FLUID SPHERE

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Abstract—A recently developed analytical technique of solving unsteady heat or mass transfer equations is applied to the case of a translating fluid sphere with the explicit purpose of scrutinizing the mathematical accuracy of the method. It is demonstrated that the technique could not only yield the surface response characteristics with high accuracy but also the details of the transient temperature or concentration field as well. Certain features of the solution, not previously recognized, are pointed out.

NOMENCLATURE

k, thermal conductivity;
p, parameter in Laplace transform;
R, radius of the fluid sphere;
t, time;
T, temperature;
U, velocity;
y, radial distance measured from the surface of the sphere;
$\operatorname{erf} x$, error function $= \frac{2}{\sqrt{\pi}} \int_0^x e^{-\beta^2} \, d\beta$;
$\operatorname{erfc} x$, complementary error function $= 1 - \operatorname{erf} x$;
$i^n\operatorname{erfc} x$, nth repeated integral of the complementary error function, $= \int_x^\infty i^{n-1} \operatorname{erfc} \beta \, d\beta, n = 1, 2, \ldots$

Greek symbols

$\Gamma\left(\frac{n}{2}\right)$, gamma function $= \int_0^\infty \beta^{(n/2)-1} e^{-\beta} \times d\beta$;
$\Gamma_x\left(\frac{n}{2}\right)$, incomplete gamma function $= \int_0^x \beta^{(n/2)-1} e^{-\beta} \, d\beta$;
θ, polar angle measured from the front stagnation;
κ, thermal diffusivity;
ξ, non-dimensional radial distance $= y/R$.

Subscripts

ss, steady state;
w, condition at the sphere surface;
∞, free stream condition;
(other symbols are defined in the text).

1. INTRODUCTION

IN A RECENT paper [1], a new analytical technique was described for solving the unsteady energy boundary layer equation for laminar flow past a flat plate. Solutions valid for all times are obtained for the surface heat flux or temperature characteristics following a sudden disturbance of the plate's thermal condition. The purpose of this communication is to further examine the usefulness of the method by applying it to seek the transient response behavior of the thermal or concentration boundary layers outside of a translating fluid sphere subsequent to a step change of its surface temperature or concentration. This case has been chosen for study mainly because of the recent availability of an exact, closed form solution for the problem [2], thus providing a unique opportunity of assessing the mathematical accuracy of the proposed technique. In addition to the surface characteristics, the instantaneous temperature or concentration profiles have also been obtained and compared with the exact solution. Such profiles were not evaluated in [1]. The new technique has the desirable feature of not only being flexible, naturally leading to solutions useful for small times but also capable of yielding accurate results.

2. ANALYSIS AND RESULTS

Figure 1 depicts a fluid sphere situated in an upflowing unbound fluid which has a uniform and constant velocity U at large distance away from it. To measure the purely mathematical satisfactoriness of the procedure, all assumptions

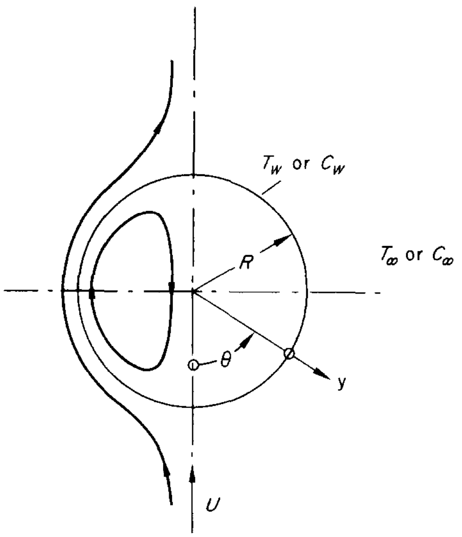

FIG. 1. Physical model and coordinate systems.

used in [2] will be adopted. It was there shown that the temperature field in the boundary layer is governed by:

$$\frac{\partial T}{\partial t} - 3U\cos\theta\frac{y}{R}\frac{\partial T}{\partial y} + \frac{3}{2}U\sin\theta\frac{1}{R}\frac{\partial T}{\partial\theta} = \kappa\frac{\partial^2 T}{\partial y^2} \qquad (1)$$

for $t > 0$ and $y > 0$. The appropriate initial and boundary conditions are

$$T(y, \theta, 0) = T_\infty \qquad (2)$$

$$T(0, \theta, t) = T_w \text{ and } T(\infty, \theta, t) = T_\infty. \qquad (3a, b)$$

Because of symmetry, the range of θ that needs to be considered is $(0, \pi)$ and the temperature field obviously must be an even function with respect to θ. In [2], both exterior and interior regions of the fluid sphere were considered. Their initial temperatures were different but uniform throughout each region. It was found that the surface temperatures of the sphere underwent a step change at the instant $t = 0^+$ but remained uniform and constant thereafter. Hence, the solution given in [2] can be directly compared with the one developed in the following sections.

For the case of mass transfer, the concentration boundary layer equation and the associated initial and boundary conditions are analogous to (1)–(3); one needs only to replace T by the solute concentration C and κ by the mass diffusivity D.

To facilitate further discussion, we introduce the following dimensionless quantities.

For Heat Transfer		*For Mass Transfer*	
$\Phi = \dfrac{T - T_\infty}{T_w - T_\infty}$	(4)	$\Phi = \dfrac{C - C_\infty}{C_w - C_\infty}$	(4′)
$\tau = \dfrac{\kappa t}{R^2}$	(5)	$\tau = \dfrac{Dt}{R^2}$	(5′)
$Pe = \dfrac{2UR}{\kappa}$	(6)	$Pe = \dfrac{2UR}{D}$	(6′)

With these, the boundary layer equation for the diffusion of either heat or mass becomes

$$\frac{\partial\Phi}{\partial\tau} - \frac{3}{2}Pe\cos\theta\xi\frac{\partial\Phi}{\partial\xi} + \frac{3}{4}Pe\sin\theta\frac{\partial\Phi}{\partial\theta} = \frac{\partial^2\Phi}{\partial\xi^2} \qquad (7)$$

with

$$\Phi(\xi, \theta, 0) = 0 \qquad (8)$$

$$\Phi(0, \theta, \tau) = 1 \text{ and } \Phi(\infty, \theta, \tau) = 0. \qquad (9a, b)$$

We define the Laplace transform of Φ in the usual way, namely,

$$\overline{\Phi}(\xi, \theta) = \int_0^\infty e^{-p\tau}\,\Phi(\xi, \theta, \tau)\,d\tau \qquad (10)$$

and obtain from (7), (8) and (9a, b)

$$\frac{\partial^2\overline{\Phi}}{\partial\xi^2} + \frac{3}{2}Pe\cos\theta\,\xi\frac{\partial\overline{\Phi}}{\partial\xi} - \frac{3}{4}Pe\sin\theta\frac{\partial\overline{\Phi}}{\partial\theta} = p\overline{\Phi} \qquad (11)$$

with

$$\overline{\Phi}(0, \theta) = p^{-1} \text{ and } \overline{\Phi}(\infty, \theta) = 0 \qquad (12a, b)$$

wherein the parametric dependence of $\overline{\Phi}$ on p is understood. Following a procedure expounded in [1], we establish that the appropriate series solution for (11) satisfying (12a, b) is

$$\overline{\Phi} = p^{-1}\{\exp[-\tfrac{3}{8}Pe\cos\theta\,\xi^2 - (p+\lambda)^{\frac{1}{2}}\xi]\} \times \sum_{n=0}^{\infty} u_n(\xi, \theta)(p+\lambda)^{-n/2} \qquad (13)$$

in which $Re(p) > 0$ and λ is a function of ξ and θ, yet unknown. It is to be evaluated from the steady state solution of the problem. Physically, λ is associated with the manner by which the local temperature decays to the steady state. The extent to which it plays a role in influencing the decay process depends also on the location in the boundary layer. In (13), we set $u_0 \equiv 1$ and $u_1(0, \theta) = u_2(0, \theta) = \ldots = 0$; hence $\overline{\Phi}(0, \theta) = p^{-1}$. We shall later demonstrate $\lambda(\xi, \theta)$ will be such that $\overline{\Phi}(\infty, \theta) = 0$.

Upon substituting (13) into (11) and equating the coefficients of like powers of $(p + \lambda)$, we find

$$\begin{aligned}
2\frac{\partial u_n}{\partial\xi} = {} & \frac{\partial^2 u_{n-1}}{\partial\xi^2} - \frac{\partial\lambda}{\partial\xi}\left[\xi\frac{\partial u_{n-2}}{\partial\xi} + (n-3)\frac{\partial u_{n-3}}{\partial\xi}\right] - \frac{3}{4}Pe\sin\theta\frac{\partial u_{n-1}}{\partial\theta} \\
& - \frac{3}{4}\left[Pe\cos\theta + \frac{3}{8}Pe^2\xi^2(1+\cos^2\theta) - \frac{4}{3}\frac{\partial}{\partial\xi}(\xi\lambda)\right]u_{n-1} \\
& - \frac{1}{2}\left[\xi\frac{\partial^2\lambda}{\partial\xi^2} - 2(n-3)\frac{\partial\lambda}{\partial\xi} - \frac{3}{4}Pe\xi\sin\theta\frac{\partial\lambda}{\partial\theta}\right]u_{n-2} \\
& - \frac{1}{2}\left[(n-3)\frac{\partial^2\lambda}{\partial\xi^2} - \frac{1}{2}\xi^2\left(\frac{\partial\lambda}{\partial\xi}\right)^2 - \frac{3}{4}(n-3)Pe\sin\theta\frac{\partial\lambda}{\partial\theta}\right]u_{n-3} \\
& + \frac{1}{2}\left(n-\frac{7}{2}\right)\xi\left(\frac{\partial\lambda}{\partial\xi}\right)^2 u_{n-4} + \frac{1}{4}(n-3)(n-5)\left(\frac{\partial\lambda}{\partial\xi}\right)^2 u_{n-5}
\end{aligned} \qquad (14)$$

for $n \geqslant 1$ and $u_{-1} = u_{-2} = u_{-3} = u_{-4} = 0$.

The recurrent relation (14) can be integrated with respect to ξ, beginning with $n = 1$, to give

$$\left.\begin{aligned}
u_1 &= -\tfrac{3}{8}Pe\xi[\mu + \tfrac{1}{8}Pe\xi^2(1+\mu^2)] + \tfrac{1}{2}\xi\lambda \\
u_2 &= \tfrac{9}{128}Pe^2\xi^2[-2 + \mu^2 + \tfrac{1}{8}Pe\xi^2\mu(1+3\mu^2) + \tfrac{1}{64}Pe^2\xi^4(1+\mu^2)^2] \\
&\quad - \tfrac{3}{16}Pe\xi^2[\mu + \tfrac{1}{8}Pe\xi^2(1+\mu^2)]\lambda + \tfrac{1}{8}\xi^2\lambda^2, \text{etc.}
\end{aligned}\right\} \qquad (15)^*$$

In (15) and others that follow, we have written μ in place of $\cos\theta$ merely for shortness. The u_n's are seen to be odd or even functions of ξ according to whether n is odd or even. The desired expression for the temperature field in the

* We have also calculated u_3, u_4 and u_5; they are omitted from the list in the interest of conserving space.

boundary layer follows immediately from inverting (13). The result is

$$\Phi(\xi, \theta, \tau) = [\exp(-\tfrac{3}{8} Pe\, \mu\xi^2 - \lambda^{\frac{1}{2}}\xi)] \sum_{n=0}^{\infty} u_n \lambda^{-n/2} G_n, \quad (16)$$

in which

$$\left.\begin{aligned} G_0 &= f_1 + f_2, \quad G_1 = f_1 - f_2 \\ G_2 &= G_0 - f_3 \\ G_3 &= G_1 + \lambda^{\frac{1}{2}}\xi f_3 - f_4 \\ G_4 &= G_0 - (1 + \lambda\tau + \tfrac{1}{2}\lambda\xi^2) f_3 + \tfrac{1}{2}\lambda^{\frac{1}{2}}\xi f_4 \\ G_5 &= G_1 + (1 + \lambda\tau + \tfrac{1}{6}\lambda\xi^2)\lambda^{\frac{1}{2}}\xi f_3 - (1 + \tfrac{2}{3}\lambda\tau + \tfrac{1}{6}\lambda\xi^2) f_4, \text{ etc.} \end{aligned}\right\} \quad (16a)$$

with

$$\left.\begin{aligned} f_1 &= \tfrac{1}{2}\operatorname{erfc}[\tfrac{1}{2}\xi\tau^{-\frac{1}{2}} - (\lambda\tau)^{\frac{1}{2}}] \\ f_2 &= \tfrac{1}{2}[\exp(2\lambda^{\frac{1}{2}}\xi)] \times \operatorname{erfc}[\tfrac{1}{2}\xi\tau^{-\frac{1}{2}} + (\lambda\tau)^{\frac{1}{2}}] \\ f_3 &= [\exp(\lambda^{\frac{1}{2}}\xi - \lambda\tau)] \operatorname{erfc}(\tfrac{1}{2}\xi\tau^{-\frac{1}{2}}) \\ &\text{and} \\ f_4 &= 2\pi^{-\frac{1}{2}}(\lambda\tau)^{\frac{1}{2}} \exp\{-[\tfrac{1}{2}\xi\tau^{-\frac{1}{2}} - (\lambda\tau)^{\frac{1}{2}}]^2\}. \end{aligned}\right\} \quad (16b)$$

It may be noted that, for $n \geqslant 2$, the G_n's are given by the following integral:

$$G_n = 2^{n-2}[\exp(\lambda^{\frac{1}{2}}\xi)] \times \int_0^{\lambda\tau} z^{(n/2)-1} e^{-z} i^{n-2} \operatorname{erfc}(\tfrac{1}{2}\lambda^{\frac{1}{2}}\xi z^{-\frac{1}{2}})\, dz \quad (16c)$$

and they are all expressible in terms of the four f-functions defined in (16b). A detailed examination of their behaviour showed that:

(i) for the entire domain of interest, namely, $0 < \tau < \infty$ and $0 < \xi < \infty$, G_n ranges from 0 to 1, and
(ii) $\lim_{\tau\to\infty} G_n = 1$.

While the proof of the foregoing results have been established only for the G_n's listed in (16a), there is strong indication that they will hold for all n's.

By letting $\tau \to \infty$ in (16), one is immediately led to the steady state temperature distribution,

$$\Phi_{ss}(\xi, \theta) = [\exp(-\tfrac{3}{8} Pe\, \mu\xi^2 - \lambda^{\frac{1}{2}}\xi)] \times \sum_{n=0}^{\infty} u_n \lambda^{-n/2}. \quad (17)$$

Differentiating (16) with respect to ξ and evaluating the result for $\xi = 0$ yield

$$-\frac{\partial\Phi}{\partial\xi}(0, \theta, \tau) = (\pi\tau)^{-\frac{1}{2}} \exp(-\lambda\tau) + \lambda^{\frac{1}{2}} \operatorname{erf}(\lambda\tau)^{\frac{1}{2}} - \sum_{n=1}^{\infty} \frac{\Gamma_{\lambda\tau}(n/2)}{\Gamma(n/2)} \lambda^{-n/2} \frac{\partial u_n}{\partial\xi}(0, \theta) \quad (18)$$

with

$$\left.\begin{aligned} &\frac{\partial u_1}{\partial\xi}(0, \theta) = -\frac{3}{8} Pe\mu + \frac{1}{2}\lambda, \qquad \frac{\partial u_2}{\partial\xi}(0, \theta) = 0 \\ &\frac{\partial u_3}{\partial\xi}(0, \theta) = -\frac{9}{128} Pe^2(2 - \mu^2) - \frac{3}{16} Pe\mu\lambda + \frac{1}{8}\lambda^2, \qquad \frac{\partial u_4}{\partial\xi}(0, \theta) = 0 \\ &\frac{\partial u_5}{\partial\xi}(0, \theta) = \frac{135}{1024} Pe^3\mu - \frac{27}{256} Pe^2(2 - \mu^2)\lambda - \frac{9}{64} Pe\mu\lambda^2 + \frac{1}{16}\lambda^3, \text{ etc.} \end{aligned}\right\} \quad (18a)$$

In (18) and (18a), λ implies $\lambda(0, \theta)$. The steady state temperature derivative at the surface of the sphere can be readily obtained from (18) by letting $\tau \to \infty$. The local transient surface flux is

$$q = -k\frac{\partial T}{\partial y}(0, \theta, t)$$
$$= -\frac{k(T_w - T_\infty)}{R}\frac{\partial \Phi}{\partial \xi}(0, \theta, \tau) \quad (19)$$

and the corresponding Nusselt number is

$$Nu = \frac{2qR}{k(T_w - T_\infty)} = -2\frac{\partial \Phi}{\partial \xi}(0, \theta, \tau). \quad (20)$$

Oftentimes in engineering analysis, the details of the transient temperature field may not be required and only the surface characteristics are of interest. If this were the case, the following short cut in the computational procedure is noteworthy. Following from the establishment of (13), one sees that $\partial\bar{\Phi}/\partial\xi\,(0, \theta)$ and its inverse $\partial\Phi/\partial\xi\,(0, \theta, \tau)$ would involve only the derivatives of the u_n functions with respect to ξ, evaluated at $\xi = 0$ since, by construction, $u_0 \equiv 1$, and $u_1(0, \theta) = u_2(0, \theta) = \ldots = 0$. Hence, all results listed in (18a) could be deduced *in succession* from the general recurrence relation (14) *without* determining the u_n's other than u_1. The observance of this procedure would result in a considerable saving of the arithmetic involved.

2.1 *Evaluation of* λ

To evaluate the function $\lambda(\xi, \theta)$, we need to separately determine the steady state solution. For the title problem such a solution has been given by Levich [3] and by Ruckenstein [4]. Rewritten in the nomenclature of this paper, it is

$$\Phi_{ss}(\xi, \theta) = \text{erfc}\left[\frac{3}{4}Pe^{\frac{1}{2}}\frac{1 + \mu}{(2 + \mu)^{\frac{1}{2}}}\xi\right]. \quad (21)$$

Thence,

$$-\frac{\partial \Phi_{ss}}{\partial \xi}(0, \theta) = \frac{3}{2}\left(\frac{Pe}{\pi}\right)^{\frac{1}{2}}\frac{1 + \mu}{(2 + \mu)^{\frac{1}{2}}} \quad (22)$$

and the steady state local Nusselt number is

$$Nu_{ss} = 3\left(\frac{Pe}{\pi}\right)^{\frac{1}{2}}\frac{1 + \mu}{(2 + \mu)^{\frac{1}{2}}}. \quad (23)$$

Here again we have written μ for cos θ. By equating (17) and (21), numerical values of $\lambda(\xi, \theta)$ could be determined for a given Peclet number. The results obtained by using terms up to and including u_5 in (17) are displayed in Fig. 2 for $\theta = 30°$, 90° and 150° and for $Pe = 500$ and 5000. Values of $\lambda(0, \theta)$ were evaluated by equating (18) with $\tau \to \infty$ and (22). For very small ξ's, the dominating terms of the series in (17) converge rapidly. However, as ξ increases, the convergence slows down. The series becomes semi-divergent with further increase in ξ and Euler's transformation was used in the evaluation of the sum. In Fig. 2, portions of the curves are

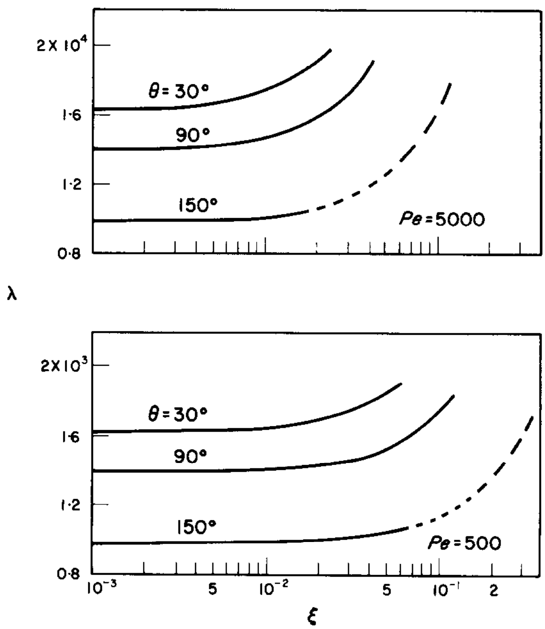

FIG. 2. Values of $\lambda(\xi, \theta)$ for $Pe = 500$ and 5000.

shown dotted; they represent the region for which the series is slowly converging and the six terms (u_0–u_5) which we have evaluated are not enough for an accurate determination of λ. The dashed curves were calculated using Euler's transformation. It is well-known that the sum of a semi-divergent series could often be obtained with good accuracy by using Euler's transformation which, however, is not suitable for slowly converging series.

The series involved in the evaluation of $\lambda(0, \theta)$ remains convergent for all cases studied and the convergence is particularly rapid for $\theta = 30°$. An inspection of the calculated data reveals that $\lambda(0, \theta)$ increases linearly with the Peclet number according to

$$\lambda(0, \theta) = C(\theta) \, . \, Pe \tag{24}$$

in which $C(\theta) = 3{\cdot}249$, $2{\cdot}778$ and $1{\cdot}958$, respectively, for $\theta = 30°, 90°$ and $150°$. This finding has some interesting implications in the surface response characteristics as we shall later see.

We now pause to remark on the behavior of the series in (16) which is not convergent when ξ is sufficiently large. From the procedure used in the determination of $\lambda(\xi, \theta)$, it can be said that series (17) necessarily possesses a limit. Since the G_n's in (16) are confined between 0 and 1, every term of the series is numerically less than the corresponding term in (17). Consequently, we have an indication that series (16) must also have a limit and that it is semi-divergent whenever it is *not* convergent. Similar remarks could be made for the series in (18) since $\Gamma_{\lambda\tau}(n/2)/\Gamma(n/2)$ is always positive and less than unity for any finite τ.

Equation (21) shows that Φ_{ss} tends to zero as $\xi \to \infty$, as it must. In view of the fact that, in general, the real part of the Laplace transform variable p must be greater than zero, it follows from (13) and (17) that $\lim_{\xi \to \infty} \bar{\Phi}(\xi, \theta) = 0$, as we have previously indicated.

2.2 *Transient temperature fields and their comparison with exact solution*

Having evaluated the λ's, the transient temperature field surrounding the fluid sphere can be calculated from (16). To effect comparison with the exact solution of [2], the time variable is reexpressed in terms of Ut/R which is $\frac{1}{2}Pe\tau$. The results are summarized in Table 1 for $Pe = 500$

Table 1. *Comparison of present analysis with exact solution of* [2]

$\theta = 30°$				$\theta = 90°$				$\theta = 150°$			
		Φ				Φ				Φ	
$\frac{Ut}{R}$	ξ	Present analysis	Exact solution	$\frac{Ut}{R}$	ξ	Present analysis	Exact solution	$\frac{Ut}{R}$	ξ	Present analysis	Exact solution
					$Pe = 500$						
	0·002	0·8208	0·8208		0·002	0·8231	0·8231		0·002	0·8253	0·8253
	0·007	0·4279	0·4279		0·007	0·4338	0·4338		0·007	0·4398	0·4398
	0·017	0·0542	0·0542		0·017	0·0573	0·0573		0·017	0·0606	0·0606
	0·027	0·0022	0·0022		0·027	0·0025	0·0025		0·027	0·0029	0·0029
0·01	0·037	0·0000	0·0000	0·01	0·047	0·0000	0·0000	0·01	0·047	0·0000	0·0000
	0·047	0·0000	0·0000		0·067	0·0000	0·0000		0·067	0·0000	0·0000
	0·057	0·0000	0·0000		0·087	0·0000	0·0000		0·097	0·0000	0·0000
	0·067	0·0000	0·0000		0·107	0·0000	0·0000		0·257	0·0000	0·0000
	0·002	0·9361	0·9361		0·002	0·9432	0·9432		0·002	0·9506	0·9506
	0·007	0·7790	0·7790		0·007	0·8031	0·8031		0·007	0·8285	0·8285
	0·017	0·4955	0·4955		0·017	0·5448	0·5448		0·017	0·5988	0·5988
0·1	0·027	0·2790	0·2790	0·1	0·027	0·3362	0·3362	0·1	0·027	0·4034	0·4034
	0·037	0·1379	0·1379		0·047	0·0941	0·0941		0·047	0·1458	0·1458
	0·047	0·0595	0·0595		0·067	0·0170	0·0170		0·067	0·0381	0·0381
	0·057	0·0223	0·0223		0·087	0·0019	0·0019		0·097	0·0027	0·0027
	0·067	0·0072	0·0072		0·107	0·0000	0·0000		0·257	0·0000	0·0000

Table 1. (*continued*)

$\theta = 30°$				$\theta = 90°$				$\theta = 150°$			
		Φ				Φ				Φ	
$\frac{Ut}{R}$	ξ	Present analysis	Exact solution	$\frac{Ut}{R}$	ξ	Present analysis	Exact solution	$\frac{Ut}{R}$	ξ	Present analysis	Exact solution
					$Pe = 500$						
1·0	0·002	0·9582	0·9582	1·0	0·002	0·9730	0·9730	1·0	0·002	0·9937	0·9940
	0·007	0·8546	0·8546		0·007	0·9058	0·9059		0·007	0·9781	0·9790
	0·017	0·6563	0·6563		0·017	0·7738	0·7741		0·017	0·9465	0·9490
	0·027	0·4797	0·4797		0·027	0·6481	0·6485		0·027	0·9157	0·9191
	0·037	0·3329	0·3327		0·047	0·4270	0·4275		0·047	0·8546	0·8597
	0·047	0·2187	0·2185		0·067	0·2577	0·2580		0·067	0·7953	0·8011
	0·057	0·1357	0·1356		0·087	0·1418	0·1419		0·097	0·7187	0·7153
	0·067	0·0795	0·0794		0·107	0·0708	0·0709		0·257	0·3270*	0·3338
					$Pe = 5000$						
0·01	0·001	0·7202	0·7202	0·01	0·001	0·7237	0·7237	0·01	0·001	0·7271	0·7271
	0·004	0·1520	0·1520		0·004	0·1573	0·1573		0·004	0·1627	0·1627
	0·007	0·0122	0·0122		0·007	0·0133	0·0133		0·007	0·0146	0·0146
	0·010	0·0003	0·0003		0·010	0·0004	0·0004		0·010	0·0005	0·0005
	0·013	0·0000	0·0000		0·013	0·0000	0·0000		0·016	0·0000	0·0000
	0·016	0·0000	0·0000		0·019	0·0000	0·0000		0·025	0·0000	0·0000
	0·019	0·0000	0·0000		0·028	0·0000	0·0000		0·031	0·0000	0·0000
	0·025	0·0000	0·0000		0·040	0·0000	0·0000		0·100	0·0000	0·0000
0·1	0·001	0·8991	0·8991	0·1	0·001	0·9103	0·9103	0·1	0·001	0·9220	0·9220
	0·004	0·6120	0·6120		0·004	0·6523	0·6523		0·004	0·6955	0·6955
	0·007	0·3748	0·3748		0·007	0·4304	0·4304		0·007	0·4933	0·4933
	0·010	0·2048	0·2048		0·010	0·2600	0·2600		0·010	0·3278	0·3278
	0·013	0·0993	0·0993		0·013	0·1431	0·1431		0·016	0·1174	0·1174
	0·016	0·0425	0·0425		0·019	0·0323	0·0323		0·025	0·0144	0·0144
	0·019	0·0160	0·0160		0·028	0·0016	0·0016		0·031	0·0024	0·0024
	0·025	0·0024	0·0024		0·040	0·0000	0·0000		0·100	0·0000	0·0000
1·0	0·001	0·9340	0·9340	1·0	0·001	0·9573	0·9574	1·0	0·001	0·9900	0·9905
	0·004	0·7405	0·7405		0·004	0·8305	0·8309		0·004	0·9602	0·9621
	0·007	0·5622	0·5622		0·007	0·7081	0·7086		0·007	0·9304	0·9337
	0·010	0·4078	0·4077		0·010	0·5928	0·5935		0·010	0·9011	0·9053
	0·013	0·2819	0·2819		0·013	0·4870	0·4877		0·016	0·8434	0·8491
	0·016	0·1854	0·1853		0·019	0·3099	0·3105		0·025	0·7612	0·7662
	0·019	0·1158	0·1157		0·028	0·1348	0·1350		0·031	0·7169	0·7124
	0·025	0·0383	0·0385		0·040	0·0328	0·0327		0·100	0·2364*	0·2343

* Euler transformation used in evaluating the sum of a series.

and 5000. When $Ut/R = 1·0$, the integrated instantaneous transfer rate over the entire sphere is within 1·5 per cent of the steady value and, thus, for all practical purposes, the steady condition would prevail. It goes without saying that the agreement is very good indeed. Finally, we note that the calculated local temperature data are not sensitive to variations in λ. For instance, when $Pe = 5000$, $\theta = 90°$ and $\xi = 0·013$, λ has been found to be $1·492 \times 10^4$ and, at $Ut/R = 1$, $\Phi = 0·4870$. If we arbitrarily increase the value of λ by 10 per cent, there results a decrease in the calculated value of Φ by approximately 1·6 per cent.

2.3 *Local heat transfer results*

The local transient heat flux at the surface of the fluid sphere has also been calculated, using values of $\lambda(0, \theta)$ given by (24). It is convenient to display the results in terms of the local Nusselt numer ratio, i.e.,

$$\frac{Nu}{Nu_{ss}} = \frac{q}{q_{ss}} = \frac{\dfrac{\partial \Phi}{\partial \xi}(0, \theta, \tau)}{\dfrac{\partial \Phi_{ss}}{\partial \xi}(0, \theta)}. \tag{25}$$

From (22), it is seen that $\partial \Phi_{ss}/\partial \xi\,(0, \theta)$ is proportional to $Pe^{\frac{1}{2}}$. Since $\lambda(0, \theta)$ varies linearly with Pe according to (24), an examination of the expression for $\partial \Phi/\partial \xi\,(0, \theta, \tau)$ reveals that the Nusselt number ratio depends on the product $Pe\tau\,(= 2\,Ut/R)$ for a given θ. In Fig. 3 the

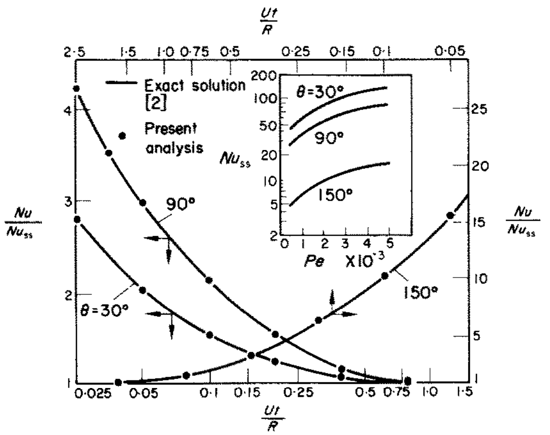

FIG. 3. Comparison of local transient surface flux calculated from the present analysis with the exact solution of [2].

presently computed data for such ratios are plotted and compared with those evaluated from the exact solution. The agreement is gratifying. For the convenience of the designer, the variation of the local steady state Nusselt number with the Peclet number is shown in the insert.

2.4 *Solution useful for small times*

Series expansions for $\exp(-\lambda\tau)$, $\text{erf}\,(\lambda\tau)^{\frac{1}{2}}$, and $\Gamma_{\lambda\tau}(n/2)$, suitable for small $\lambda\tau$, are well-known. Upon substituting them into (18), combining terms and rearranging, followed by a division with (22), there is obtained

$$\frac{Nu}{Nu_{ss}} = \frac{\sqrt{2}}{3}\frac{(2+\mu)^{\frac{1}{2}}}{1+\mu} X^{-\frac{1}{2}}\big[1 + \tfrac{3}{2}\mu X + \tfrac{3}{8}(2-\mu^2)X^2 - \tfrac{9}{16}\mu X^3 + 0(X^4)\big] \tag{26}$$

in which $X \equiv Ut/R$. It is interesting to note that $\lambda(0, \theta)$ does not appear in (26). An analogous expression for the thermal response behavior of the laminar boundary layer over a flat plate exhibits a similar character [1].

The first term of the series in (26) represents the conduction transient as one would expect. By comparing data calculated from (26) with those from equation (45) of [2] which is mathematically exact, the following errors have been noted. At $X = 0{\cdot}4$, (26) shows errors of 0·12 per cent, 0·74 per cent and 1·4 per cent, respectively, for $\theta = 30°$, 90° and 150°. At smaller X, the errors are uniformly less; however, they grow rapidly with increasing X, particularly at large θ.

We have also evaluated the integrated instantaneous heat transfer rate over the entire sphere and compared it with the exact solution. Excellent agreement is again observed.

3. CONCLUDING REMARKS

Since the recent introduction of the new analytical technique for examining the thermal response behavior of laminar boundary layer flows as described in [1], there is the urgent need of assessing the mathematical accuracy of the method. This communication fulfills, in part, such need.

For the case of heat or mass transfer from a translating fluid sphere, it is demonstrated that, in addition to the surface characteristics, the details of the transient temperature (or concentration) field can also be obtained by the method and with high accuracy. This was not attempted in [1]. In this respect, it is pertinent to point out that, should the information on the transient temperature field be desired for the two problems examined in the said reference, the λ's must be considered as a function of the similarity variable η, which was defined by equation (8) of that reference. Clearly, all numerical values of λ reported therein are actually $\lambda(0)$.

In view of the apparent flexibility of the method and its capability of yielding highly accurate results, further development is currently in progress in our laboratory. It is hoped that non-similar flows can likewise be treated as well as certain turbulent boundary layers.

REFERENCES

1. B. T. CHAO and L. S. CHEEMA, Unsteady heat transfer in laminar boundary layer over a flate plate, *Int. J. Heat Mass Transfer* **11**, 1311–1324 (1968).
2. B. T. CHAO, Transient heat and mass transfer to a translating droplet, *J. Heat Transfer* **91**, 273–281 (1969).
3. V. G. LEVICH, *Physicochemical Hydrodynamics*, p. 406. Prentic-Hall, New Jersey (1962). A factor of $\sqrt{(3/2)}$ is inadvertently left out in the numerator of the expression for N, p. 407.
4. E. RUCKENSTEIN, On heat transfer between vapour bubbles in motion and the boiling liquid from which they are generated, *Chem. Engng Sci.* **10**, 22–30 (1959). Equation (12) of this paper carries an extraneous factor $2/\sqrt{\pi}$ which should be deleted.

SOLUTION SOUS FORME DE SÉRIE DU TRANSPORT DE CHALEUR OU DE MASSE INSTATIONNAIRE VERS UNE SPHÈRE FLUIDE EN TRANSLATION

Résumé—Une technique analytique récemment développée pour résoudre les équations instationnaires de transport de chaleur ou de masse est appliquée au cas d'une sphère fluide en translation dans le but explicite d'examiner à fond la précision mathématique de la méthode. On démontre que la technique pourrait non seulement fournir les caractéristiques de la réponse superficielle avec une précision élevée, mais aussi les détails du champ des températures ou des concentrations transitoires. Certaines caractéristiques de la solution, no reconnues auparavant, sont indiquées.

REIHEN-LÖSUNG DES INSTATIONÄREN WÄRME- ODER STOFFAUSTAUSCHES AN EINER BEWEGTEN FLÜSSIGKEITSKUGEL

Zusammenfassung— Eine kürzlich entwickelte analytische Methode zur Lösung der Gleichungen für den Wärme- oder Stoffaustausch wird auf den Fall einer bewegten Flüssigkeitskugel angewandt, mit der besonderen Absicht, die mathematische Genauigkeit dieser Methode zu prüfen. Es wird gezeigt, dass die Methode nicht nur die charakteristischen Werte an der Oberfläche mit hoher Genauigkeit liefern kann, sondern ebenso die Einzelheiten der instationären Temperatur—oder Konzentrationsverteilung. Auf gewisse, bisher unerkannte Eigenschaften dieser Lösung, wird besonders hingewiesen.

АСИМПТОТИЧЕСКОЕ РЕШЕНИЕ ТЕПЛО-ИЛИ МАССОПЕРЕНОСА К ПЕРЕМЕЩАЮЩЕЙСЯ ЖИДКОЙ СФЕРЕ

Аннотация—Недавно разработанная аналитическая методика решения уравнении нестационарного тепло-или массопереноса применена к случаю перемещающейся сферы с целью исследования математической точности метода. Показано, что методика не только может дать с большой точностью частотные характеристики поверхности, но и подробные значения неустановившейся температуры, а также значение поля концентрации. Отмечаются некоторые особенности решения не упоминавшиеся ранее.